

# QUANTUM GROTHENDIECK POLYNOMIALS

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## ABSTRACT

We study the algebraic aspects of (small) quantum equivariant  $K$ -theory of flag manifold. Lascoux-Schützenberger's type formula for quantum double and quantum double dual Grothendieck polynomials and the quantum Cauchy identity for quantum Grothendieck polynomials are obtained.

## §1. Introduction.

There exists a remarkable hierarchy of polynomials related with the generalized cohomology theories of the flag manifold:

- double and double dual Grothendieck polynomials  $G_w^{(\beta)}(x, y)$  and  $\mathcal{H}_w^{(\beta)}(x, y)$ , corresponding to the equivariant  $K$ -theory;
- double Schubert polynomials  $\mathfrak{S}_w(x, y) = G_w^{(\beta)}(x, y)|_{\beta=0}$ , corresponding to the equivariant cohomology theory;
- Schubert polynomials  $\mathfrak{S}_w(x) = \mathfrak{S}_w(x, y)|_{y=0}$ , corresponding to the singular cohomology theory.

The Grothendieck polynomials, which are the  $K$ -theoretic analog of Schubert polynomials, was introduced by Alexander Grothendieck in his study of the general Riemann-Roch Theorem [G]. Algebraic definition and fundamental properties of Grothendieck and Schubert polynomials were introduced and developed in series of Alain Lascoux and Marcel-Paul Schützenberger papers, see, for example, [LS], [L], for more references see [M].

The hierarchy, mentioned above, reflects the well-known fact, that the corresponding generalized cohomology theories are the only ones which have a polynomial group law.

In this paper we continue the quantization of some interesting classes of polynomial and introduce the quantum double,  $\tilde{G}_w(x, y)$ , and quantum double dual,  $\tilde{\mathcal{H}}_w(x, y)$ , Grothendieck polynomials and investigate their properties.

Quantum double Schubert polynomials,  $\tilde{\mathfrak{S}}_w(x, y)$ , were introduced and their properties were studied in [KM].

Quantum Schubert polynomials,  $\mathfrak{S}_w^q(x)$ , were introduced and their properties were studied in [FGP]. Another approach to quantum Schubert polynomials is considered in [KM].

The structure of this paper is the following. In §2 some results on double Schubert and double Grothendieck polynomials, quantum analogs of which we would like to construct,

are collected. Probably, the formulae (5),(6),(7) was not appear in earlier publications. In §3 we recall some results from [FGP], which will be used in §4. In §4 we introduce quantum double and quantum double dual Grothendieck polynomials and prove the quantum Cauchy identity.

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## §2. Classical Grothendieck polynomials.

In this section we give a brief review of the theory of Grothendieck and Schubert polynomials created by A. Lascoux and M.P.-Schützenberger, see e.g. [L], [LS] and [M]. A proof of Cauchy's formula for Grothendieck polynomials can be found in [FK], and that of Pieri's rule for double Grothendieck polynomials can be obtained using the generalization of method from [KV].

### 2.1. Divided and isobaric divided differencies.

Let  $x_1, \dots, x_n$  and  $\beta$  be independent set of variables, and let  $P_n := P_{n,\beta} = \mathbf{Z}[x_1, \dots, x_n, \beta]$  for each  $n \geq 1$ .

Let us denote by  $\Lambda_n := \Lambda_{n,\beta} = \mathbf{Z}[\beta][x_1, \dots, x_n]^{S_n} \subset P_n$  the subring of symmetric polynomials in  $x_1, \dots, x_n$  with polynomial coefficients and by

$$H_n := H_{n,\beta} = \left\{ \sum_{I=(i_1, \dots, i_n)} a_I x^I \mid a_I \in \mathbf{Z}[\beta], 0 \leq i_k \leq n-k, \forall k \right\}$$

the additive subgroup of  $P_n$  spanned by all monomials  $a_I x^I$  with  $a_I \in \mathbf{Z}[\beta]$  and  $I \subset \delta := \delta_n = (n-1, n-2, \dots, 1, 0)$ .

For  $i \leq n-1$  let us define a  $\Lambda_n$ -linear operator  $\partial_i$  acting on  $P_n$

$$(\partial_i f)(x) = \frac{f(x_1, \dots, x_i, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_{i+1}, x_i, \dots, x_n)}{x_i - x_{i+1}}$$

Divided difference operators  $\partial_i$  satisfy the following relations

$$\begin{aligned} \partial_i^2 &= 0, \\ \partial_i \partial_j &= \partial_j \partial_i, \text{ if } |i-j| \geq 2, \\ \partial_i \partial_{i+1} \partial_i &= \partial_{i+1} \partial_i \partial_{i+1}, \end{aligned} \tag{1}$$

and the Leibnitz rule

$$\partial_i(fg) = \partial_i(f)g + s_i(f)\partial_i(g).$$

Let us also introduce the operators  $\pi_i := \pi_i^+$  and  $\pi_i^-$  ( $1 \leq i \leq n-1$ ) defined by

$$(\pi_i^\pm f)(x) = (\partial_i f)(x) \pm \beta \partial_i(x_{i+1} f(x)).$$

In place of (1) we have

$$\begin{aligned} (\pi_i^\pm)^2 &= \pm\beta\pi_i, \\ \pi_i\pi_j &= \pi_j\pi_i, \text{ if } |i-j| \geq 2, \\ \pi_i\pi_{i+1}\pi_i &= \pi_{i+1}\pi_i\pi_{i+1}, \end{aligned}$$

and the modified Leibnitz rule

$$\pi_i(fg) = \pi_i(f)g + s_i(f)(\pi_i + \beta)(g),$$

It is clear that if  $f \in \Lambda_n$ , then  $(\pi_i + \beta)(f) = 0$ , and  $\pi_i(fg) = f\pi_i(g)$ .

For any permutation  $w \in S_n$ . let us denote by  $R(w)$  the set of reduced decompositions for  $w$ , i.e. sequences  $(a_1, \dots, a_p)$  such that  $w = s_{a_1} \dots s_{a_p}$ , where  $p = l(w)$  is the length of the permutation  $w \in S_n$ , and  $s_i = (i, i+1)$  is the simple transposition that interchanges  $i$  and  $i+1$ .

For any sequence  $\mathbf{a} = (a_1, \dots, a_p)$  of positive integers, we define

$$\partial_{\mathbf{a}} = \partial_{a_1} \dots \partial_{a_p}, \quad \pi_{\mathbf{a}} = \pi_{a_1} \dots \pi_{a_p}.$$

**Proposition 1** ([M], (2.5), (2.6), (2.15)). *If  $\mathbf{a}, \mathbf{b} \in R(w)$ , then  $\partial_{\mathbf{a}} = \partial_{\mathbf{b}}$  and  $\pi_{\mathbf{a}} = \pi_{\mathbf{b}}$ .*

If  $\mathbf{a}$  is not reduced, then  $\partial_{\mathbf{a}} = 0$ . It follows from Proposition 1, that the operators  $\partial_w = \partial_{\mathbf{a}}$  and  $\pi_w = \pi_{\mathbf{a}}$  are well-defined, where  $\mathbf{a}$  is any reduced word for  $w$ .

## 2.2. Double Grothendieck and double dual Grothendieck polynomials.

Let  $x = (x_1, \dots, x_n)$ ,  $y = (y_1, \dots, y_n)$  be two sets of independent variables and

$$G_{w_0}(x, y) := \prod_{i+j \leq n} (x_i + y_i).$$

**Definition 1** (Lascoux-Schützenberger [LS],[L], see also [FK]). *For each permutation  $w \in S_n$  the double Grothendieck polynomial  $G_w(x, y)$  and double dual Grothendieck polynomial  $\mathcal{H}_w(x, y)$  are defined to be*

$$\begin{aligned} G_w(x, y) &= \pi_{w^{-1}w_0} G_{w_0}(x, y), \\ \mathcal{H}_w(x, y) &= \psi_{w^{-1}w_0} G_{w_0}(x, y), \end{aligned}$$

where  $w_0$  is the longest element of  $S_n$ , and  $\psi_w := \sum_{v \leq w} \beta^{l(w)-l(v)} \pi_v$ .

**Proposition 2** ([L], [LS]).

- (Stability) Let  $m > n$  and let  $i : S_n \hookrightarrow S_m$  be the embedding. Then

$$G_w(x, y) = G_{i(w)}(x, y).$$

- $\pi_w = \sum_{v \leq w} (-\beta)^{l(w)-l(v)} \psi_v$ , and hence,

$$\mathcal{H}_w(x, y) = \sum_{w \leq v} \beta^{l(v)-l(w)} G_v(x, y)$$

$$G_w(x, y) = \sum_{w \leq v} (-\beta)^{l(v)-l(w)} \mathcal{H}_v(x, y).$$

- Grothendieck and dual Grothendieck polynomials  $G_w(x) := G_w(x, 0)$  and  $\mathcal{H}_w(x) := \mathcal{H}_w(x, 0)$  from the  $\mathbf{Z}[\beta]$ -bases of  $H_n$ .

- (Cauchy formula, [FK])

$$\sum_{w \in S_n} \mathcal{H}_w(x, \ominus z) G_{w w_0}(y, z) = \prod_{i+j \leq n} (x_i + y_i + \beta x_i y_i), \quad (2)$$

where  $\ominus z := \left( \frac{-z_1}{1 - \beta z_1}, \frac{-z_2}{1 - \beta z_2}, \dots, \frac{-z_n}{1 - \beta z_n} \right)$ .

- $\mathcal{H}_w^{(\beta)}(x, y) = G_{w^{-1}}^{(-\beta)}(y, x)$ ,  $w \in S_n$ .

Hence,  $(\pi_{w w_0})^{(y)} G_{w w_0}(x, y) = G_{w^{-1}}^{(-\beta)}(y, x) = \mathcal{H}_w^{(\beta)}(x, y)$ .

**Examples.** 1) Double Grothendieck and double dual Grothendieck polynomials for  $S_3$

$$\begin{aligned} G_{121}(x, y) &= (x_1 + y_1)(x_1 + y_2)(x_2 + y_1), \\ G_{12}(x, y) &= (x_1 + y_1)(x_2 + y_1)(1 - \beta y_2), \\ G_{21}(x, y) &= (x_1 + y_1)(x_1 + y_2)(1 - \beta y_1), \\ G_1(x, y) &= (x_1 + y_1)(1 - \beta y_1)(1 - \beta y_2), \\ G_2(x, y) &= (x_1 + x_2 + y_1 + y_2 + \beta x_1 x_2 - \beta y_1 y_2)(1 - \beta y_1), \\ G_{\text{id}}(x, y) &= (1 - \beta y_1)^2(1 - \beta y_2); \\ \mathcal{H}_{121}(x, y) &= G_{121}(x, y), \\ \mathcal{H}_{12}(x, y) &= (x_1 + y_1)(x_2 + y_1)(1 + \beta x_1), \\ \mathcal{H}_{21}(x, y) &= (x_1 + y_1)(x_1 + y_2)(1 + \beta x_2), \\ \mathcal{H}_1(x, y) &= (x_1 + y_1)(1 + \beta x_1)(1 + \beta x_2), \\ \mathcal{H}_2(x, y) &= (x_1 + x_2 + y_1 + y_2 + \beta x_1 x_2 - \beta y_1 y_2)(1 + \beta x_1), \\ \mathcal{H}_{\text{id}}(x, y) &= (1 + \beta x_1)^2(1 + \beta x_2). \end{aligned}$$

2) Let  $w \in S_n$  be a dominant permutation with code  $c(w) = (c_1 \geq c_2 \geq \dots \geq c_n \geq 0)$ . Then

$$\frac{G_w(x, y)}{G_{\text{id}}(x, y)} = \prod_{k=1}^n \prod_{i=1}^{c_k} \frac{x_k + y_i}{1 - \beta y_i}.$$

### 2.3. Scalar product.

Let us define a scalar product on  $P_{n,\beta}$  with values in  $\Lambda_{n,\beta}$ , by the rule

$$\langle\langle f, g \rangle\rangle = \pi_{w_0}(f, g), \quad f, g \in P_{n,\beta},$$

where  $w_0$  is the longest element in  $S_n$ . The scalar product  $\langle\langle, \rangle\rangle$  defines a non-degenerate pairing  $\langle\langle, \rangle\rangle_0$  on the quotient ring  $P_{n,\beta}/I_{n,\beta}$ , where  $I_n := I_{n,\beta}$  is the ideal in  $P_{n,\beta}$  generated by the elementary symmetric polynomials  $e_1(x), \dots, e_n(x)$ .

#### Proposition 3.

- $\langle\langle \pi_w f, g \rangle\rangle = \langle\langle f, \pi_{w^{-1}} g \rangle\rangle$ ,  $f, g \in P_{n,\beta}$ .
- $\langle\langle f, G_w \rangle\rangle_0 = \eta(\pi_{w_0 w} f)$ , where  $\eta : P_{n,\beta} \rightarrow \mathbf{Z}[\beta]$  is the homomorphism defined by  $\eta(1) = 1$ ,  $\eta(x_i) = 0$ ,  $1 \leq i \leq n$ .
- (Interpolation formula) If  $f \in H_n$ , then

$$f(x)G_{\text{id}}(x, -y) = \sum_{w \in S_n} \mathcal{H}_w(x, -y) \pi_w^{(y)} f(y). \quad (3)$$

- (Orthogonality, [LS], [L]) Let  $u, v \in S_n$ , then

$$\langle\langle \mathcal{H}_u(x), G_v(x) \rangle\rangle_0 = \begin{cases} 1, & \text{if } v = w_0 u, \\ 0, & \text{otherwise} \end{cases}$$

As for every  $w \in S_n$ ,  $\langle\langle G_w(x, y), 1 \rangle\rangle = \pi_{w_0}(G_w(x, y)) = (-\beta)^{l(w_0 w)} G_{\text{id}}(x, y)$ , the fact that

$$\langle\langle \mathcal{H}_w(x, y), 1 \rangle\rangle = \begin{cases} G_{\text{id}}(x, y), & \text{if } w = w_0, \\ 0, & \text{otherwise} \end{cases}$$

generalizes the property of the Moebius function (for the Ehresman-Bruhat order on  $S_n$ ) to be equal to  $\pm 1$ , cf. [LS].

**Remark.** We have

$$G_{\text{id}}(x, y) = \prod_{k=1}^{n-1} (1 - \beta y_k)^{n-k},$$

$$\mathcal{H}_{\text{id}}(x, y) = \prod_{k=1}^{n-1} (1 + \beta x_k)^{n-k}.$$

### 2.4. Pieri's rule for Grothendieck polynomials.

Let us denote by  $(i, j)$  the transposition that interchanges  $i$  and  $j$ ,  $i < j$ .

**Proposition 5** ([FL]).

$$G_{s_k}(x) \cdot G_w(x) \equiv \sum_v \beta^{l(v)-l(w)-1} G_v(x) \pmod{I_n}, \quad (4)$$

where sum runs over  $v \in S_n$ , such that

- i)  $v = w \cdot (i_1, j_1) \dots (i_{m+1}, j_{m+1})$ ,
- ii)  $i_l \leq k < j_l$  for all  $l = 1, 2, \dots, m+1$ ,
- iii)  $l(v) = l(w) + m + 1$ .

More generally, consider the ring

$$P_{n,\beta}[y] := \mathbf{Z}[\beta][x_1, \dots, x_n, y_1, \dots, y_n]$$

and the ideal  $J_n := J_{n,\beta}$  in  $R_{n,\beta}[y]$  generated by the following polynomials

$$e_i(x) + (-1)^{i-1} e_i(y), \quad i = 1, \dots, n.$$

**Proposition 6** (Pieri's rule for double Grothendieck polynomials).

$$G_{s_k}(x, y_w) \cdot G_w(x, y) \equiv G_{\text{id}}(x, y_w) \left[ \sum_v \beta^{l(v)-l(w)-1} G_v(x, y) \right] \pmod{J_n}, \quad (5)$$

where  $y_w = (y_{w(1)}, \dots, y_{w(n)})$ , and the sum runs over the same set of permutations  $v \in S_n$  as in Proposition 5.

## 2.5. Canonical involution $\omega$ .

There exists an involution  $\omega$  of the ring  $P_{n,\beta}[y]$  given by  $\omega(x) = \overleftarrow{x}$ ,  $\omega(y) = \overleftarrow{y}$ , where for any sequence  $z = (z_1, \dots, z_m)$  we define  $\overleftarrow{z}$  to be equal to  $(z_m, z_{m-1}, \dots, z_2, z_1)$ . It is clear that involution  $w$  preserves the ideals  $I_n$  and  $J_n$ .

**Proposition 7.** *Let  $v$  be a permutation, then*

$$\omega(G_v(x, y)) \mathcal{H}_{\text{id}}(x, y) \equiv (-1)^{l(v)} \mathcal{H}_{w_0 v w_0}(x, y) \omega(G_{\text{id}}(x, y)) \pmod{J_{n,\beta}}. \quad (6)$$

**Remark.** Let  $G_w^{LS}(A, B)$  be the double Grothendieck polynomials introduced by A. Lascoux and M.P.-Schützenberger in [L], (2.1). Then

$$G_w^{LS}(1 - \beta y, 1 + \beta x) = (-1)^{l(w)} \frac{G_{w^{-1}}(x, y)}{G_{\text{id}}(x, y)}.$$

## 2.6. Double Schubert polynomials.

Let us remind the definition of double Schubert polynomials due to A. Lascoux and M.P.-Schützenberger.

**Definition 2** (A. Lascoux– M.P.-Schützenberger). *For each permutation  $w \in S_n$ , the double Schubert polynomial  $\mathfrak{S}_w(x, y)$  is defined to be*

$$\mathfrak{S}_w(x, y) = \partial_{w^{-1}w_0}^{(x)} \mathfrak{S}_{w_0}(x, y),$$

where divided difference operator  $\partial_{w^{-1}w_0}^{(x)}$  acts on the  $x$  variables, and

$$\mathfrak{S}_{w_0}(x, y) = G_{w_0}(x, y) = \prod_{i+j \leq n} (x_i + y_j).$$

It follows from Definition 1 and 2 that

$$G_w(x, y)|_{\beta=0} = \mathcal{H}_w(x, y)|_{\beta=0} = \mathfrak{S}_w(x, y).$$

Double Schubert polynomials appear in algebra and geometry as cohomology classes related to degeneracy loci of flagged vector bundles. If  $h : E \rightarrow F$  is a map of rank  $n$  vector bundles on a smooth variety  $X$ ,

$$E_1 \subset E_2 \subset \cdots \subset E_n = E, \quad F := F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1$$

are flags of subbundles and quotient bundles, then there is a degeneracy locus  $\Omega_w(h)$  for each permutation  $w$  in the symmetric group  $S_n$ , described by the conditions

$$\Omega_w(h) = \{x \in X \mid \text{rank}(E_p(x) \rightarrow F_q(x)) \leq \#\{i \leq q, w_i \leq p\}, \forall p, q\}.$$

For generic  $h$ ,  $\Omega_w(h)$  is irreducible,  $\text{codim} \Omega_w(h) = l(w)$ , and the class  $[\Omega_w(h)]$  of this locus in the Chow ring of  $X$  is equal to the double Schubert polynomial  $\mathfrak{S}_{w_0w}(x, -y)$ , where

$$\begin{aligned} x_i &= c_1(\ker(F_i \rightarrow F_{i-1})), \\ y_i &= c_1(E_i/E_{i-1}), \quad 1 \leq i \leq n. \end{aligned}$$

It is well-known [F1] that the Chow ring of flag variety  $Fl_n$  admits the following description

$$CH^*(Fl_n) \cong \mathbf{Z}[x_1, \dots, x_n, y_1, \dots, y_n]/J,$$

where  $J$  is the ideal generated by

$$e_i(x_1, \dots, x_n) - e_i(y_1, \dots, y_n), \quad 1 \leq i \leq n,$$

and  $e_i(x)$  is the  $i$ -th elementary symmetric function in the variables  $x_1, \dots, x_n$ .

Result below describes the structure of quotient ring  $\mathbf{Z}[x, y]/J$ .

**Proposition** ([LS], [KV]). *The ring  $\mathbf{Z}[x_1, \dots, x_n, y_1, \dots, y_n]/J_n$  is a free module of dimension  $n!$  over the ring  $R$ , with basis either  $\mathfrak{S}_w(x)$ , or  $\mathfrak{S}_w(x, y)$ ,  $w \in S_n$ , where*

$$R := \frac{\mathbf{Z}[x_1, \dots, x_n] \otimes \mathbf{Sym}[y_1, \dots, y_n]}{J}.$$

### 2.7. Chern classes.

Let us denote by  $1 + P_n^+$  the multiplicative monoid of polynomials in two sets of variables  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  with rational coefficients and constant term 1. There exists a homomorphism (the Chern homomorphism)

$$c : \mathbf{Z}[x^{\pm 1}, y^{\pm 1}] \rightarrow 1 + P_n^+$$

such that  $c(1) = 1$ ,  $c(A + B) = c(A)c(B)$ ,  $A, B \in \mathbf{Z}[x^{\pm 1}, y^{\pm 1}]$ . On the basis  $x^I y^J$ ,  $I \in \mathbf{Z}^n$ ,  $J \in \mathbf{Z}^n$ , it takes the values

$$c(x^I y^J) = 1 + \sum_{k=1}^n i_k x_k - \sum_{k=1}^n j_k y_k.$$

**Proposition 8** (cf. [LS], §5). *Let us assume that  $\beta = -1$ . Then*

$$\begin{aligned} \bullet c(y^{-\delta} G_v(1 - x, y - 1)) &\equiv 1 - \sum_{v \leq w} a_{vw} \mathfrak{S}_w(x, y) \pmod{J_n}, \\ \bullet c(x^{-\delta} \mathcal{H}_v(1 - x, y - 1)) &\equiv 1 - \sum_{v \leq w} b_{vw} \mathfrak{S}_w(x, y) \pmod{J_n}, \end{aligned} \tag{7}$$

where  $a_{vv} = b_{vv} = (-1)^{l(v)}(l(v) - 1)!$ .

### §3. Quantization map.

In this section we describe a remarkable construction of quantization map, which is due to S. Fomin, S. Gelfand and A. Postnikov [FGP], see also [KM], where some additional properties of quantization map are given. In [KM] the quantization map was constructed using the Interpolation formula, [M], (6.8), by replacing the classical double Schubert polynomials by their quantum analogs. Let us remind a construction from [FGP] in the form most convenient for our purposes. The starting point is a remarkable family of commuting operators

$$X_j := X_j^{(n)} = x_j - \sum_{i < j} q_{ij} \partial_{(ij)} + \sum_{j < k} q_{jk} \partial_{(jk)}, \quad 1 \leq j \leq n,$$

where (for  $i < j$ )  $\partial_{(ij)} = \partial_{t_{ij}} = \partial_i \partial_{i+1} \dots \partial_{j-1} \dots \partial_{i+1} \partial_i$  is the divided difference operator corresponding to the transposition  $t_{ij}$ , and  $q_{ij} = q_i q_{i+1} \dots q_{j-1}$ . It is clear that  $X_i : P_n \rightarrow P_n$ , but the following property is less obvious.

**Proposition** ([FGP], Theorem 3.1). *The operators  $X_j$  commute pairwise.*

Another useful property of operators  $X_i$  is the following.



**Lemma** ([FGP], Lemma 3.3). *For any polynomial  $f \in P_n$ , there exists a unique operator  $F \in \mathbf{Z}[q_1, \dots, q_{n-1}][X_1, \dots, X_n]$  such that*

$$F(X_1, \dots, X_n)(1) = f(x_1, \dots, x_n).$$

The map  $f \mapsto F : P_n \rightarrow P_n$  is called the quantization map, [FGP], Section 3. Proposition below describes one of the main property of quantization map.

**Proposition** ([FGP], (5.1)). *Let  $\tilde{\mathfrak{S}}_w(x)$  be the quantum Schubert polynomial corresponding to a permutation  $w \in S_n$ . Then*

$$\tilde{\mathfrak{S}}_w(X_1, \dots, X_n)(1) = \mathfrak{S}_w(x_1, x_2, \dots, x_n). \quad (9)$$

The relation (9) is very useful and allows to reduce the proofs of statements about quantum Schubert polynomials to those about classical ones, see e.g. [FGP], proof of Theorem 7.8. However, it seems rather difficult to find an explicit (e.g. of Lascoux-Schützenberger's type) formula for a quantum Schubert polynomial  $\tilde{\mathfrak{S}}_w(x)$  using only the relation (9). So we came to the problem how to describe effectively the quantization map. To solve this problem, in the paper [FGP], Section 4, the basis of standard elementary monomials are used. However, this basis does not give an orthogonal basis with respect to the canonical pairing on the ring of polynomials  $\mathbf{Z}[x_1, \dots, x_n]$ , see e.g. [LS]; [M], (5.2). In the paper [KM] we gave another approach to the quantization problem which is based on the theory of quantum double Schubert polynomials and quantum Cauchy's identity. Our main observations are:

i) with respect to the  $y$  variables, the quantum double Schubert polynomials have a structure which is very similar to that in the classical case;

ii) the quantum Cauchy identity, in essential, is equivalent to the orthogonality of quantum Schubert polynomials with respect to quantum pairing, see [KM], (2.5).

Quantum double Schubert polynomials  $\tilde{\mathfrak{S}}_w(x, y)$ , were defined in [KM] by using the Lascoux-Schützenberger type construction. Namely, by Definition 6, [KM],

$$\tilde{\mathfrak{S}}_w(x, y) = \partial_{w^{-1}w_0}^{(y)} \tilde{\mathfrak{S}}_{w_0}(x, y),$$

where

$$\tilde{\mathfrak{S}}_{w_0}(x, y) = \Delta_1(y_{n-1}|x_1)\Delta_2(y_{n-1}|x_1, x_2) \dots \Delta_{n-1}(y_1|x_1, \dots, x_{n-1})$$

and  $\Delta_k(t|x_1, \dots, x_k)$  is the Givental-Kim determinant [GK]:

$$\Delta_k(t|x) := \det \begin{pmatrix} x_1 + t & q_1 & 0 & \dots & \dots & \dots & 0 \\ -1 & x_2 + t & q_2 & 0 & \dots & \dots & 0 \\ 0 & -1 & x_3 + t & q_3 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1 & x_{k-2} + t & q_{k-2} & 0 \\ 0 & \dots & \dots & 0 & -1 & x_{k-1} + t & q_{k-1} \\ 0 & \dots & \dots & \dots & 0 & -1 & x_k + t \end{pmatrix} \quad (2)$$

$$= \sum_{i=1}^k t^{k-i} \tilde{e}_i(x_1, \dots, x_k \mid q_1, \dots, q_{k-1}).$$

The polynomials  $\tilde{e}_i(x) := \tilde{e}_i(x_1, \dots, x_k \mid q_1, \dots, q_{k-1})$  are called the quantum elementary symmetric polynomials degree  $i$  in the variables  $x_1, \dots, x_k$ .

The main goal of this paper is to apply the results from [FGP] in the context of [KM]. In sequel, we will use the notations from [KM]; e.g. we denote by  $\tilde{f}$  the quantization of polynomial  $f$ .

The following result will be used in §4.

**Theorem 1.** *Let  $w_0$  be the longest element in  $S_n$ , then*

$$\tilde{\mathfrak{S}}_{w_0}(X_1, \dots, X_n, y_1, \dots, y_n)(1) = \mathfrak{S}_{w_0}(x, y).$$

*Proof.* We use two facts from [FGP], Corollary 4.6 and Proposition 4.10:

- $\tilde{e}_i(X_1, \dots, X_n)(1) = e_i(x_1, \dots, x_n)$ ,
- $\tilde{f}g = f\tilde{g}$ , if  $f \in \Lambda_n$ .

Thus, we have  $\Delta_{n-1}(y_1 \mid X_1, \dots, X_{n-1})(1) =$

$$= \sum_k y_1^{n-1-k} \tilde{e}_k(X_1, \dots, X_{n-1})(1) = \sum_k y_1^{n-1-k} e_k(x_1, \dots, x_{n-1}) = \prod_{k=1}^{n-1} (y_1 + x_k).$$

Hence,

$$\begin{aligned} \tilde{\mathfrak{S}}_{w_0}(X_1, \dots, X_{n-1}, y_1, \dots, y_{n-1})(1) &= \Delta_1(y_{n-1} \mid X_1^{(n)}) \cdots \Delta_{n-1}(y_1 \mid X_1^{(n)}, \dots, X_{n-1}^{(n)})(1) \\ &= \Delta_1(y_{n-1} \mid X_1^{(n-1)}) \cdots \Delta_{n-2}(y_2 \mid X_1^{(n-1)}, \dots, X_{n-2}^{(n-1)})(1) \prod_{k=1}^{n-1} (y_1 + x_k) = \prod_{i=1}^{n-1} \prod_{k=1}^{n-i} (y_i + x_k). \end{aligned}$$

■

#### §4. Quantum double and quantum double dual Grothendieck polynomials.

Let  $x = (x_1, \dots, x_n)$ ,  $y = (y_1, \dots, y_n)$  be two sets of variables, and (cf. [KM])

$$\tilde{\mathfrak{S}}_{w_0}(x, y) := \tilde{\mathfrak{S}}_{w_0}^{(q)}(x, y) = \prod_{i=1}^{n-1} \Delta_i(y_{n-i} \mid x_1, \dots, x_i),$$

where  $\Delta_k(t \mid x_1, \dots, x_k) := \sum_{j=0}^k t^{k-j} e_j(x_1, \dots, x_k \mid q_1, \dots, q_{k-1})$  is the Givental-Kim determinant (2), which is generating function for the quantum elementary symmetric functions in the variables  $x_1, \dots, x_k$ .

**Definition 3.** For each permutation  $w \in S_n$ , the quantum double dual Grothendieck polynomial  $\tilde{\mathcal{H}}_w(x, y)$  is defined to be

$$\tilde{\mathcal{H}}_w(x, y) = (\pi_{w w_0}^-)^{(y)} \tilde{\mathfrak{S}}_{w_0}(x, y).$$

Here symbol  $(\pi_{w w_0}^-)^{(y)}$  means that isobaric operator  $\pi_{w w_0}^-$  acts on the  $y$  variables.

**Corollary 1** (of Theorem 1). for each permutation  $w \in S_n$ ,

$$\begin{aligned} \tilde{\mathfrak{S}}_w(X, y)(1) &= \mathfrak{S}_w(x, y), \\ \tilde{\mathcal{H}}_w(X, y)(1) &= \mathcal{H}_w(x, y). \end{aligned} \tag{10}$$

**Remarks.** 1) In the “classical limit”  $q_1 = \cdots = q_{n-1} = 0$ ,

$$\tilde{\mathcal{H}}_w(x, y)|_{q=0} = (\pi_{w w_0}^-)^{(y)} G_{w_0}(x, y) = G_{w^{-1}}^{(-\beta)}(y, x) = \mathcal{H}_w^{(\beta)}(x, y),$$

i.e.  $\mathcal{H}_w(x, y)|_{q=0} = \mathcal{H}_w^{(\beta)}(x, y)$ .

2) (Stability) Let  $m > n$  and let  $i : S_n \hookrightarrow S_m$  be the embedding. Then

$$\tilde{\mathcal{H}}_w(x, y) = \tilde{\mathcal{H}}_{i(w)}(x, y).$$

Let us define the quantum dual Grothendieck polynomial  $\tilde{\mathcal{H}}_w(x)$  as the specialization

$$\tilde{\mathcal{H}}_w(x) := \tilde{\mathcal{H}}_w(x, y)|_{y=0}.$$

**Definition 4.** For each permutation  $w \in S_n$ , the quantum double Grothendieck polynomial  $\tilde{G}_w(x, y)$  is defined to be

$$\tilde{G}_w(x, y) = \sum_{w \leq v} (-\beta)^{l(v)-l(w)} \tilde{\mathcal{H}}_w(x, y).$$

Let us define the quantum Grothendieck polynomials  $\tilde{G}_w(x)$  as the specialization

$$\tilde{G}_w(x) := \tilde{G}_w(x, y)|_{y=0}.$$

**Example.** Quantum double and quantum double dual Grothendieck polynomials for  $S_3$ .

$$\begin{aligned} \tilde{G}_{121}(x, y) &= (x_1 + y_1)(x_1 + y_2)(x_2 + y_1) + q_1(x_1 + y_2) \\ \tilde{G}_{12}(x, y) &= [(x_1 + y_1)(x_2 + y_1) + q_1](1 - \beta y_2), \\ \tilde{G}_{21}(x, y) &= [(x_1 + y_1)(x_1 + y_2) - q_1](1 - \beta y_1), \\ \tilde{G}_1(x, y) &= (x_1 + y_1)[(1 - \beta y_1)(1 - \beta y_2) + q_1 \beta^2], \end{aligned}$$

$$\begin{aligned}
\tilde{G}_2(x, y) &= (x_1 + x_2 + y_1 + y_2 - \beta y_1 y_2 + \beta x_1 x_2 + \beta q_1)(1 - \beta y_1), \\
\tilde{G}_{\text{id}}(x, y) &= (1 - \beta y_1)[(1 - \beta y_1)(1 - \beta y_2) + q_1 \beta^2]; \\
\tilde{\mathcal{H}}_{121}(x, y) &= \tilde{G}_{121}(x, y), \\
\tilde{\mathcal{H}}_{12}(x, y) &= [(x_1 + y_1)(x_2 + y_1) + q_1](1 + \beta x_1), \\
\tilde{\mathcal{H}}_{21}(x, y) &= [(x_1 + y_1)(x_2 + y_2) - q_1](1 + \beta x_2) + q_1 \beta (x_1 + x_2 + y_1 + y_2), \\
\tilde{\mathcal{H}}_1 &= (x_1 + y_1)[(1 + \beta x_1)(1 + \beta x_2) + q_1 \beta^2], \\
\tilde{\mathcal{H}}_2 &= (x_1 + x_2 + y_1 + y_2 + \beta x_1 x_2 - \beta y_1 y_2 + \beta q_1)(1 + \beta x_1), \\
\tilde{\mathcal{H}}_{\text{id}} &= (1 + \beta x_1)[(1 + \beta x_1)(1 + \beta x_2) + q_1 \beta^2].
\end{aligned}$$

**Remark.** One can show that

$$\begin{aligned}
(\pi_{w_0}^-)^{(y)} \tilde{G}_{w_0}(x, y) &:= \tilde{G}_{\text{id}}(x, y) = \beta^{\frac{n(n-1)}{2}} \tilde{G}_{w_0}(x, (\beta^{-1}, \beta^{-1}, \dots, \beta^{-1})), \\
\tilde{\mathcal{H}}_{\text{id}}^{(\beta)}(x, y) &= \tilde{G}_{\text{id}}^{(-\beta)}(y, x).
\end{aligned}$$

**Theorem 2.** (Quantum Cauchy identity for Grothendieck polynomials).

$$\sum_{w \in S_n} \tilde{\mathcal{H}}_w(x, \ominus z) G_{ww_0}(y, z) = \prod_{k=1}^{n-1} \Delta_k^{(\beta)}(y_{n-k} \mid x_1, \dots, x_k), \quad (11)$$

where  $\Delta_k^{(\beta)}(t \mid x_1, \dots, x_k) = (1 + \beta t)^k \Delta_k\left(\frac{t}{1 + \beta t} \mid x_1, \dots, x_k\right)$ .

Proof follows from the classical Cauchy identity for Grothendieck polynomials (2), Theorem 1 and (10).

Let us denote the product in the RHS(11) by  $\tilde{\mathbf{G}}_{w_0}(x, y)$ .

**Corollary 2** (of Theorem 2).

- $\tilde{\mathcal{H}}_w(x) = \psi_{ww_0}^{(y)} \tilde{\mathbf{G}}_{w_0}(x, y)|_{y=0}$ ,
- $\tilde{G}_w(x) = \pi_{ww_0}^{(y)} \tilde{\mathbf{G}}_{w_0}(x, y)|_{y=0}$ ,
- $\tilde{G}_w(x) = \sum_{v \in S_n} \eta(\pi_{ww_0} G_{vw_0}(x)) \tilde{\mathcal{H}}_v(x)$ .

It seems interesting to study the properties of polynomials  $\tilde{\mathbf{G}}_w(x, y) = \pi_{ww_0}^{(y)} \tilde{\mathbf{G}}_{w_0}(x, y)$  and  $\tilde{\mathbf{H}}_w(x, y) = \psi_{ww_0}^{(y)} \tilde{\mathbf{G}}_{w_0}(x, y)$ .

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